MTH 305: Practice assignment 6

1 Number-theoretic functions

Establish the following assertions.

(i) Let m and n be positive integers and $p_1, p_2, \ldots p_r$ be the distinct primes that divide atleast one of m or n. Then m and n can be written in the form
$$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}, \quad \text{with } k_i \geq 0 \text{ for } i = 1, 2, \ldots, r.$$
$$n = p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}, \quad \text{with } j_i \geq 0 \text{ for } i = 1, 2, \ldots, r.$$
Then $\gcd(m, n) = 1$ if and only if $k_i j_i = 0$ for $i = 1, 2, \ldots, r$.

(ii) For any integer $n \geq 1$, $\tau(n) \leq 2\sqrt{n}$.

(iii) If $n > 1$ is a composite number, then $\sigma(n) > n + \sqrt{n}$.

(iv) Given a positive integer $k > 1$, show that there are infinitely many integers n for which $\tau(n) = k$, but at most finitely many n with $\sigma(n) = k$.

(v) There is no positive integer n satisfying $\sigma(n) = 10$.

(vi) Let f and g be multiplicative functions that are not identically zero and have the property that $f(p^k) = g(p^k)$ for each prime p and $k \geq 1$, then $f = g$.

(vii) For any positive integer n, $\sum_{d|n} \tau(d)^3 = (\sum_{d|n} \tau(d))^2$.

(viii) For any positive integer n, $\sum_{d|n} (n/d)\sigma(d) = \sum_{d|n} d\tau(d)$.

(ix) For any positive integer n, $\sum_{d|n} 1/d = \sigma(n)/n$.

2 The Möbius Inversion Formula

Establish the following assertions.

(i) $\Lambda(n) = \sum_{d|n} \mu(n/d) \log d = -\sum_{d|n} \mu(d) \log d$, where

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k, \text{ where } p \text{ is a prime and } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(ii) If $n > 1$ has prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, then:

(a) $\sum_{d|n} \mu(d)\sigma(d) = (-1)^r p_1 p_2 \cdots p_r$.

(b) $\sum_{d|n} d\mu(d) = (1 - p_1)(1 - p_2) \cdots (1 - p_r)$.

(iii)

$$S(n) = \sum_{d|n} |\mu(d)| = 2^{\omega(n)},$$

where $S(n)$ denote the number of square-free divisors of n and $\omega(n)$ is the number of distinct prime divisors of n.

(iv) If $n > 1$ has prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, then:

(a) $\sum_{d|n} \mu(d)\lambda(d) = 2^{\omega(n)}$.

(b) $\sum_{d|n} \lambda(n/d) 2^{\omega(d)} = 1$.

Here, λ is a function defined as $\lambda(1) = 1$ and $\lambda(n) = (-1)^{k_1+k_2+\cdots+k_r}$.

3 The Greatest Integer Function

Establish the following assertions.

(i) Given integers a and $b > 0$, show that there exists a unique integer r with $0 \leq r < b$ satisfying $a = [a/b]b + r$.

(ii) If x and y are real numbers, then:

(a) $[x] + [-x] = 0$ or -1.

(b) $[x/n] = [[x]/n]$ for any positive integer n.

(c) $[x] + [y] + [x + y] \leq [2x] + [2y]$.

(iii) If $n \geq 1$ and p is a prime, show that $(2n)!/(n!)^2$ is an even integer.

(iv) Let n be positive integer is written as $n = a_k p^k + \cdots + a_2 p^2 + a_1 p + a_0$, where p is a prime and $0 \leq a_i < p$. Then the exponent of the highest power of p appearing in the prime factorization of $n!$ is

$$\frac{n - (a_k + \cdots + a_2 + a_1 + a_0)}{p - 1}$$

(v) For a positive integer N, we have:

(a) $\sum_{n=1}^{N} \mu(n)[N/n] = 1$.

(b) $|\sum_{n=1}^{N} \mu(n)/n| \leq 1$.

(c) $N = \sum_{n=1}^{2N} \tau(n) - \sum_{n=1}^{N}[2N/n]$.

(d) $\tau(N) = \sum_{n=1}^{N}([N/n] - [(N-1)/n])$.

4 Euler's Phi-Function

Establish the following assertions.

(i) $\phi(3n) = 3\phi(n)$ if and only if $3 \mid n$.

(ii) $\phi(3n) = 2\phi(n)$ if and only if $3 \nmid n$.

(iii) For any positive integer n, $\frac{1}{2}\sqrt{n} \leq \phi(n) \leq n$.

(iv) For $n > 1$ composite number, $\phi(n) \leq n - \sqrt{n}$.

(v) If the integer n has r distinct odd prime factors, then $2^r \mid \phi(n)$.

(vi) If every prime that divides n also divides m, then $\phi(nm) = n\phi(m)$ for every positive integer n.

(vii) If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, then:

(a) $\sigma(n)\phi(n) \geq n^2(1 - 1/p_1^2)(1 - 1/p_2^2)\cdots(1 - 1/p_r^2)$.

(b) $\tau(n)\phi(n) \geq n$.

(viii) If $d \mid n$, show that $\phi(d) \mid \phi(n)$.

(ix) For positive integers m and n, and $d = \gcd(m, n)$, we have:

   (a) $\phi(m)\phi(n) = \phi(mn)\dfrac{\phi(d)}{d}$.

   (b) $\phi(m)\phi(n) = \phi(\gcd(m, n))\phi(\operatorname{lcm}(m, n))$.

(x) There is no integer n for which $\phi(n) = n/4$.

(xi) For a positive integer k, there are at most finite number of integers n for which $\phi(n) = k$.

(xii) If p is a prime and $k \geq 2$, show that

$$\phi(\phi(p^k)) = p^{k-2}\phi((p-1)^2).$$

5 Euler's Theorem

Establish the following assertions.

(i) $2^{15} - 2^3 \mid a^{15} - a^3$, for any integer a.

(ii) If $\gcd(a, n) = \gcd(a - 1, n) = 1$, then

$$1 + a + a^2 + \cdots + a^{\phi(n)-1} \equiv 0 \pmod{n}.$$

(iii) If m and n are relatively prime positive integers, then

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}.$$

(iv) If $\gcd(a, n) = 1$, then the linear congruence $ax \equiv b \pmod{n}$ has the solution $x \equiv ba^{\phi(n)-1} \pmod{n}$.

(v) For any integer a, a and a^{4n+1} have the same last digit.

(vi) For any prime p, we have:

   (a) $\tau(p!) = 2\tau((p-1)!)$.

   (b) $\sigma(p!) = (p+1)\sigma((p-1)!)$.

(c) $\phi(p!) = (p+1)\phi((p-1)!)$.

(vii) For positive integer n,

$$\sum_{d|n}(-1)^{n/d}\phi(d) = \begin{cases} 0, & \text{if } n \text{ is even, and} \\ -n, & \text{if } n \text{ is odd.} \end{cases}$$

(viii) For positive integer n, $\sum_{d|n} \mu^2(d)/\phi(d) = n/\phi(n)$.

(ix) For positive integer n, $\sum_{d=1}^{n} \phi(d)[n/d] = n(n+1)/2$.

(x) If n is a square-free integer, we have:

(a) $\sum_{d|n} \sigma(d^{k-1})\phi(d) = n^k$ for integers $k \geq 2$.

(b) $\tau(n^2) = n$ if and only if $n = 3$.

(xi) For $n > 2$, $\phi(n^2) + \phi((n+1)^2) \leq 2n^2$.

(xii) If $n = p(p+2)$ is a product of twin primes, then $\phi(n)\sigma(n) = (n+1)(n-3)$.

(xiii) If $a_1, a_2, \ldots, a_{\phi(n)}$ is a reduced set of residues modulo n, then

$$a_1 + a_2 + \cdots + a_{\phi(n)} \equiv 0 \pmod{n}, \qquad \text{for } n > 2.$$